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EXCITED RANDOM WALK IN TWO DIMENSIONS HAS LINEAR SPEED

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Imagine a cookie placed on every vertex of an infinite d-dimensional grid. A random walker on this grid behaves as follows upon encountering a cookie: he consumes it and then performs a random step with an ϵ-drift to the right, namely the probability to make a right move is $\frac{1}{2d}+\epsilon$, the probability for a left move is $\frac{1}{2d}-\epsilon$ and the probability for all other moves is $\frac{1}{2d}$. When encountering a site already visited (so no cookie), he performs a simple random walk. This process was coined by Benjamini and Wilson [BW03] "excited random walk" (though the name "brownie walk" might describe it better). Since then a number of papers were devoted to this process. See [V03, K, Z, ABK], [PW97, D99] for a one dimensional continuous version and [AR05] for some simulation results.

[BW03] mostly discusses the geometric case $d>1$. They prove that in dimension ≥ 4 the walk has linear speed, namely

$$\varliminf_{n\to\infty}\frac{E(n)_1}{n}>0 \tag{1}$$

where $E(n)$ is the position of the walk at time n and $E(n)_1$ is its first (left-right) coordinate. In dimension 2 they prove transience, in fact they prove that $E(n)_1 > cn^{3/4}\log^{-5/4}n$ for n sufficiently large almost surely, and ask what is the correct speed. [K] extended (1) to the three dimensional case. The purpose of this paper is to show the same for two dimensions (in one dimension this is not true, though the multiple cookies case discussed by Zerner [Z] is still open).

Benjamini and Wilson's proof of transience in two dimensions will play a crucial role in the current paper so let's describe it briefly. They coupled excited random walk to a simple random walk R in the natural way: when the ERW encounters a cookie they walk "as close as possible", that is with probability 2ϵ the ERW walks to the right and the SRW walks to the left, while with probability $1-2\epsilon$ they perform the same step. If no cookie, they just perform the same step. This means that (with this coupling) the ERW is always to the right of the SRW and the distance is increasing. This implies that when the SRW reaches a tan point, a point (x,y) such that the random walk never visited before $(x+n,y)$ for any $n=0,1,2,\ldots$ then the ERW must be at a vertex with a cookie. The name "tan point" comes from placing the sun at right infinity, and these points are the points when the SRW can tan since its past path is not blocking the sun. Hence an estimate of the number of tan points gives a lower bound for the number of cookies eaten.

Thus we are left with a problem on simple random walk, i.e. estimate the probability that $R(n)$ is a tan point. By symmetry this is the same as the probability that a random walk of length n will avoid hitting a half line. This, as is well known

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is $\approx n^{-1/4}$ — see [K87]. The same $\frac{1}{4}$ translates to the $\frac{3}{4}$ exponent in the speed estimate of Benjamini and Wilson with some logarithmic corrections.

To get better than $n^{3/4}$ one needs to apply ERW's "self-correcting" property. Very roughly, if an ERW of length m goes to a distance of μ for $\mu \gg \sqrt{m}$ then the next portion of length m of the ERW should be quite independent from the previous portion, and should also continue to a length of μ. Unfortunately, one cannot continue this forever and claim that the speed is $> \mu/m$ because there is always a small probability for a portion to fail and then one needs a "fallback mechanism". This is provided by the Benjamini and Wilson argument (performed locally, see the definition of a relative tan point below). Thus the proof is inductive, using Benjamini and Wilson to both kickstart the induction and to provide a fallback mechanism in each stage.

I wish to thank Itai Benjamini and Martin Zerner for enlightening discussions of this problem.

1. Preliminaries

Definition. Let $\epsilon < \frac{1}{4}$. Let $\mathcal{V} \subset \mathbb{Z}^2$ be some subset of vertices ($\mathcal{V}$ standing for "visited") and let $w \in \mathbb{Z}^2$. ϵ-Excited random starting from $(\mathcal{V}, w)$ is a stochastic process $E(n)$ on $\mathbb{Z}^2$ such that $E(0) = w$ and such that if $E(n) = E(m)$ for some $m < n$ or $E(n) \in \mathcal{V}$ then $E(n+1)$ has probability $\frac{1}{4}+\epsilon$ to be $E(n)+(1,0)$, probability $\frac{1}{4}-\epsilon$ to be $E(n)+(-1,0)$ and probability $\frac{1}{4}$ to be either $E(n)+(0,\pm 1)$. In the other case, that $E(n) \notin E[0, n-1] \cup \mathcal{V}$, distribute $E(n+1)$ like a simple random walk.

Let us next state the result with an explicit bound on the probability:

Theorem. *Let E be an ϵ-excited random walk starting from $(\emptyset, (0,0))$. Then with probability $> 1 - Ce^{-c\log^2 n}$ one has $E(n)_1 > cn$. The constants C and c may depend on ϵ.*

For a simple or excited walk R and two times $i \le j$ we shall denote by $R[i,j]$ the set $\{R(k) : i \le k \le j\} \subset \mathbb{Z}^2$. $\log x$ will always be a shortcut for $\max\{\log x, 1\}$. $\lfloor x \rfloor$ and $\lceil x \rceil$ will denote, as usual, the largest integer $\le x$ and the smallest integer $\ge x$ respectively. By C and c we shall denote constants depending only on ϵ whose precise value is unimportant as far as this paper is concerned, and could change from formula to formula or even within the same formula. C will pertain to constants which are "big enough" and c to constants which are "small enough". We will number a few C and c-s — only those which we will reference later on. When we say "x is sufficiently large" we mean "$x > C$ for some C" and in particular the bound may depend on ϵ.

2. Proof

Definition. Let R be a simple random walk and let $i < j$ be two times. Then we say that SRW has a **tan point** at j **relative** to i if the portion of the walk $R[i, j-1]$ does not intersect the half line $R(j) + \mathbb{N} \times \{0\}$.

Lemma 1. *Let R be a simple random walk of length n, and let $m < n$ be sufficiently large. Then with probability $\ge 1 - Ce^{-c\log^2 n}$ the following holds: for every $0 \le i < n - m\log^6 n$ the random walk on $\left[i, i + \left\lfloor m \log^6 n \right\rfloor\right]$ exhibits $\ge c_1 m^{3/4}$ tan points relative to i in some interval $[j, j+m]$ for $i + m \le j \le i + m\log^6 n - m$.*

Proof. Fix i. Denote by B_1 (B standing for "bad") the event that

$$\max_{j\in[i,i+m]} |R(j)_2 - R(i)_2| > \tfrac{1}{2}\log^2 n\sqrt{m}.$$

It is well known that $\mathbb{P}(B_1) \le Ce^{-c\log^4 n}$ — for each j this follows from the Chernoff bound and summing over j only changes the constant in the exponent. Next denote by B_2 the event

$$\max\left\{|R(j)_2 - R(i)_2| : j \in \left[i, i + \left\lfloor m\log^6 n\right\rfloor - m\right]\right\} < 2\log^2 n\sqrt{m}.$$

Again, it is well known that $\mathbb{P}(B_2) \le Ce^{-c\log^2 n}$ — for any $x \in I$,

$$I := \left[\left\lceil R(i)_2 - 2\log^2 n\sqrt{m}\right\rceil, \left\lfloor R(i)_2 + 2\log^2 n\sqrt{m}\right\rfloor\right]$$

there is a probability $> c$ to exit I in the next $\left\lfloor m\log^4 n\right\rfloor$ steps (if m is sufficiently large), and so the probability to exit it in $\left\lfloor m\log^6 n\right\rfloor - m$ steps is $\ge 1 - Ce^{-c\log^2 n}$.

Let λ be some parameter to be fixed later and assume $m > 1/\lambda$. Let $k \in \mathbb{Z}\setminus\{0\}$ be positive or negative and denote by L_k, S_k and H_k the horizontal line, strip and half strip at height $R(i)_2 + k\left\lfloor\sqrt{\lambda m}\right\rfloor$ respectively. In a formula:

$$\begin{aligned}
L_k &:= \mathbb{Z}\times\left\{R(i)_2 + k\left\lfloor\sqrt{\lambda m}\right\rfloor\right\}\\
S_k &:= \mathbb{Z}\times\left(R(i)_2 + \left](k-1)\left\lfloor\sqrt{\lambda m}\right\rfloor, (k+1)\left\lfloor\sqrt{\lambda m}\right\rfloor\right[\right)\\
H_k &:= \{v \in S_k : |v_2| \ge |R(i)_2|\}.
\end{aligned}$$

Denote by T_k the first time (after i) when R hits L_k (for the purpose of the definition of T_k we extend the walk to infinity). Next denote by $T_k^* > T_k$ the first time after T_k when R exits the strip S_k. Examine now the event G_k that R has $\ge c(\lambda m)^{3/4}$ relative tan points in the half strip H_k in the time interval $[T_k, T_k^*]$. Translation and reflection symmetry shows that the G_k are i.i.d. It is known that $\mathbb{P}(G_k) \ge c_2$, see [BW03]. Further, if λ is sufficiently small then $\mathbb{P}(T_k^* - T_k > m) \le \frac{1}{2}c_2$ uniformly in $m > 1/\lambda$. Fix λ to satisfy this requirement and denote $G_k^* = G_k \cap \{T_k^* - T_k \le m\}$ so that $\mathbb{P}(G_k^*) \ge c$.

The events G_k^* are also independent so if we denote by $B_3^\pm$ the event

$$\neg G_k^* \text{ for all } k \in \pm\left]\frac{\log^2 n}{\sqrt{\lambda}}, 2\frac{\log^2 n}{\sqrt{\lambda}} - 1\right[$$

then $\mathbb{P}(B_3^\pm) \le Ce^{-c\log^2 n}$ (here we consider λ as a constant and allow C and c to depend on it). The lemma is now finished since if none of the four bad events B_1, B_2, B_3^+, B_3^- happened the claim holds. Indeed, $\neg B_2$ implies that either

$$T_{\left\lfloor 2\lambda^{-1/2}\log^2 n\right\rfloor} < i + m\log^6 n - m \quad\text{or}\quad T_{-\left\lfloor 2\lambda^{-1/2}\log^2 n\right\rfloor} < i + m\log^6 n - m.$$

Assume that the first happened. Then if B_3^+ did not happen then some G_k^* happened and by the definition of G_k^* we can denote $j := T_k^*$ and get what we want ($\neg B_1$ is used to show $j \ge i+m$). Hence with probability $\ge 1 - Ce^{-c\log^2 n}$ we found a j for our i. Summing over i we are done. □

Lemma 2. *With probability $\ge 1 - Ce^{-c\log^2 n}$ one has that for any $i \ne j$, $|R(j)_1 - R(i)_1| \le \log n\sqrt{i-j}$.*

This follows immediately from the Chernoff bound.

Lemma 3. *Let E be an excited random walk of length $2n$ starting from some $\mathcal{V} \subset \left]-\infty, -n^{5/8}\right] \times \mathbb{Z}$ and some vertex $E(0) \in \mathbb{Z}^2$. Let $m \geq n^{15/16}$. Then with probability $\geq 1 - C_1 e^{-c\log^2 n}$ one has that either*

(i) *$E(n)_1 < 0$; or*
(ii) *For every $n \leq i \leq 2n - m\log^6 2n$,*

$$E\big(i + \left\lfloor m \log^6 2n \right\rfloor\big)_1 - \max_{j \leq i} E(j)_1 \geq cm^{3/4}.$$

Proof. We may assume n is sufficiently large (for n small choosing C_1 sufficiently large will render the lemma true trivially). If $m > n\log^{-6} 2n$ then (ii) holds vacuously so assume the opposite. Couple E to a simple random walk R as in the introduction. Let B_1 and B_2 be the bad events of lemmas 1 and 2 for the walk R, i.e. B_1 is the event that for some $i \in \{0, \ldots, \left\lfloor 2n - m\log^6 2n \right\rfloor\}$ there aren't enough relative tan points and B_2 is the event that for some $i \neq j$ $|R(i)_1 - R(j)_1|$ is very large. Let T_k be the k-th time when E reached a new vertex and let ξ_k be $E(T_{k+1}) - E(T_k) - (R(T_{k+1}) - R(T_k))$. ξ_k is a vector but since it can take only the values $(0, 0)$ and $(2, 0)$ we will consider it as a scalar. Let B_3 be the event that for some $k \leq 2n$,

$$\sum_{l=k}^{k+\left\lfloor c_1 m^{3/4} \right\rfloor} \xi_l \leq c_3 m^{3/4}.$$

Now, the ξ_k-s are independent hence it is easy to see that for c_3 sufficiently small for any k the probability for this is $< Ce^{-cm^{3/4}}$ and then so will be their sum. In other words, $\mathbb{P}(B_3) < Ce^{-cm^{3/4}}$.

The lemma would be proved if we show that $\neg(B_1 \cup B_2 \cup B_3 \cup \text{(i)}) \Rightarrow \text{(ii)}$. This will be done inductively, and the first step is to choose a number $1 \leq i_1 \leq n$ as follows. We divide into two cases according to whether $E(0)_1 > -\frac{1}{2}n^{5/8}$ or not. If $E(0)_1 > -\frac{1}{2}n^{5/8}$ we choose $i_1 = 0$. Otherwise we note that if $E(0)_1 \leq -\frac{1}{2}n^{5/8}$ and $E(n)_1 > 0$ then we can choose some $i_1 \in [0, \ldots, n]$ with the properties that $E\left(i_1\right)_1 > -\frac{1}{2}n^{5/8}$ and $E\left(i_1 + m\right)_1 - E\left(i_1\right)_1 > \frac{1}{4}mn^{-3/8} \geq \frac{1}{4}n^{9/16}$ for n sufficiently large.

Now, by $\neg B_2$ we see that for all $j \leq i_1$ one has $R(j)_1 \leq R\left(i_1\right)_1 + \log 2n\sqrt{2n}$ and the coupling implies the same for E. Similarly we get that for all $j \geq i_1 + m$, $E(j)_1 \geq E\left(i_1 + m\right)_1 - \log 2n\sqrt{2n}$. Hence we see that (for n sufficiently large)

$$(\mathcal{V} \cup E[1, i_1]) \cap E[i_1 + m, 2n] = \emptyset.$$

This means that any tan point of R relative to i_1 after $i_1 + m$ is a point where E encountered a new vertex. By $\neg B_1$ we know that for some $i_2 \in [i_1 + m, i_1 + m\log^6 n - m]$ we would have $\geq c_1 m^{3/4}$ relative tan points in $[i_2, i_2 + m]$ and hence $\geq c_1 m^{3/4}$ new points for E. By $\neg B_3$ we see that

$$E\left(i_2 + m\right)_1 - E\left(i_2\right)_1 > c_3 m^{3/4} + R\left(i_2 + m\right)_1 - R\left(i_2\right)_1$$

and by $\neg B_2$

$$\geq c_3 m^{3/4} - \log 2n\sqrt{2n} > cm^{3/4} > \tfrac{1}{4}n^{9/16}$$

for n sufficiently large.

Now we can repeat the argument of the last paragraph with i_1 replaced by i_2. We get a sequence of i-s satisfying (except possibly i_1)

$$E(i_j + m) - E(i_j) > cm^{3/4}$$

and $i_j \le i_{j-1} + m\log^6 n - m$. This implies (again with $\neg B_2$) (ii) and the lemma. $\square$

Lemma 4. *Let $B_1, \dots, B_n$ be i.i.d. Bernoulli (i.e. 0-1) variables with $\mathbb{P}(B_i = 1) = \epsilon$. Then $\mathbb{P}(\sum B_i > k) \le 2(n\epsilon)^k$.*

This is a straightforward calculation (and a very rough estimate to boot — we will only use it for $\epsilon \ll 1/n$ where it is rather close to the truth).

Lemma 5. *Let $E, n, \mathcal{V}$ and m be as in lemma 3 and assume in addition that $m \ge 2n^{15/16}\log^6 2n$. Assume also that one knows that an excited random walk of length $2m$ starting from any $\mathcal{W} \subset \left]-\infty, -m^{5/8}\right] \times \mathbb{Z}$ and any vertex in $\mathbb{Z}^2$ satisfies, with probability $\ge 1-\epsilon$, that either*

$$E(m)_1 < 0\text{; or}$$
$$E(2m)_1 - E(m)_1 > \mu.$$

(here end the assumptions of the lemma). Then with probability $\ge 1 - Ce^{-c\log^2 n} - 4(n\epsilon)^{\lfloor \log n\rfloor}$, either

$$E(n)_1 < 0\text{; or} \tag{2}$$
$$E(2n)_1 - E(n)_1 > \mu(\lfloor n/m\rfloor - 2\log n - 1). \tag{3}$$

Proof. We may assume n is sufficiently large. Let $k \in \{0, \dots, \lfloor n/m\rfloor - 2\}$ and let B^k be the event that

$$E(n+(k+1)m)_1 > \max\left\{\max_{j\le n+km} E(j)_1, \max_{v\in\mathcal{V}} v_1\right\} + m^{5/8}$$
$$E(n+(k+2)m)_1 - E(n+(k+1)m) \le \mu.$$

We translate by $-E(n+(k+1)m)$ and get from the assumption that

$$\mathbb{P}(B^k \mid E[0, \dots, n+km]) \le \epsilon.$$

Since $B^0, \dots, B^{k-2}$ depend only on $E[0, \dots, km]$ then we get that the even B^k-s are stochastically dominated by a sequence of i.i.d. ϵ-Bernoulli variables. The odd B^k satisfy the same. Hence by lemma 4,

$$\mathbb{P}(\#B^k > 2\log n) \le 4(\lceil \lfloor n/m\rfloor/2\rceil \epsilon)^{\lfloor\log n\rfloor} < 4(n\epsilon)^{\lfloor \log n\rfloor}.$$

Denote this event by B_1.

We now apply lemma 3 with $m_{\text{lemma 3}} = l = \lfloor m\log^{-6} n\rfloor$ (and the same n). If n is sufficiently large then $l \ge n^{15/16}$ and lemma 3 may indeed be applied. We get that with probability $> 1 - Ce^{-c\log^2 n}$ we have either (2) or for every $n \le i \le 2n - l\log^6 2n$,

$$E\left(i + \left\lfloor l\log^6 2n\right\rfloor\right)_1 - \max_{j\le i} E(j)_1 \ge cl^{3/4}. \tag{4}$$

Since $\lfloor l\log^6 2n\rfloor \simeq m$ (the difference is $\le \log^6 2n + 1$) we can replace the first by the second in (4) and pay only in the constant. Denote therefore by B_2 the event that $\neg$(2) and also

$$\exists i \in \{n, \dots, 2n-m\} : E(i+m)_1 - \max_{j\le i} E(j)_1 < c_4 l^{3/4}.$$

Then for c_4 sufficiently small we have $\mathbb{P}(B_2) \le Ce^{-c\log^2 n}$.

We claim that $\neg(B_1 \cup B_2 \cup (2)) \Rightarrow (3)$, which will finish the lemma. This however, is clear: $E(n)_1 \ge 0$ and $\neg B_2$ give that for all $k \in \{0, \ldots, \lfloor n/m \rfloor - 2\}$,

$$E(n+(k+1)m)_1 \ge \max\left\{\max_{j\le n+km} E(j)_1, \max_{v\in\mathcal{V}} v_1\right\} + cl^{3/4}$$

so for n sufficiently large (so that $cl^{3/4} > m^{5/8}$) $\neg B^k$ will give

$$E(n+(k+2)m)_1 > E(n+(k+1)m) + \mu.$$

Only $\lfloor 2\log n \rfloor$ B^k-s are allowed to fail, so

$$\sum_{k:\neg B^k} E(n+(k+2)m)_1 - E(n+(k+1)m)_1 > \mu(\lfloor n/m \rfloor - 2\log n - 1).$$

While if B^k does not hold we can still use $\neg B_2$ to get

$$E(n+(k+2)m)_1 - E(n+(k+1)m)_1 > 0$$

and we are done. □

Lemma 6. *With the notations of lemma 3, with probability* $\ge 1 - Ce^{-c\log^2 n}$*, either*

(i) $E(n)_1 < 0$; *or*
(ii) $E(2n)_1 - E(n)_1 > cn$.

Proof. This follows by an inductive application of lemma 5, but one has to be careful with the parameters. Let therefore K and k be two parameters which will be fixed later. Define $\alpha_n = \alpha_n(K,k)$ as the maximal number such that

$$\mathbb{P}(\{E(n)_1 > 0\} \cap \{E(2n)_1 - E(n)_1 \le \alpha_n n\}) \le Ke^{-k\log^2 n} \quad \forall \mathcal{V}, E(0)$$

(as usual we assume $\mathcal{V} \subset \left]-\infty, -n^{5/8}\right] \times \mathbb{Z}$). We wish to estimate α_n. First we check what lemma 3 has to say about α_n. Choosing $m = \left\lfloor n\log^{-6} 2n \right\rfloor$ (which can be done if n is sufficiently large) and $i = n$ we get that with probability $> 1 - Ce^{-c\log^2 n}$, either $E(n)_1 < 0$ or

$$\begin{aligned} E(2n)_1 - E(n)_1 &\ge E\left(n + \left\lfloor m\log^6 2n \right\rfloor\right)_1 - E(n)_1 - \log^6 2n - 1 \ge \\ &\ge cm^{3/4} - \log^6 2n - 1 > cn^{5/8}. \end{aligned}$$

As usual we can remove the assumption that n is sufficiently large and pay only in the constants. In other words, if K is sufficiently large and k is sufficiently small then $\alpha_n \ge c(K,k)n^{-3/8}$ for all n.

Next we translate lemma 5 to α_n notations and it now goes: if we have

$$Ce^{-c\log^2 n} + 4\left(nKe^{-k\log^2 m}\right)^{\lfloor \log n \rfloor} \le Ke^{-k\log^2 n} \tag{5}$$

then

$$\alpha_n \ge \alpha_m \left(1 - \frac{(2\log n - 2)m}{n}\right).$$

It is easy to see that for K sufficiently large and k sufficiently small (5) will hold for n sufficiently large (all bounds depend only on the C and c in (5)). Fix K and k to satisfy both requirements. We get that for n sufficiently large,

$$\alpha_n \ge \alpha_m \left(1 - n^{-1/32}\right) \quad m = \left\lceil 2n^{15/16}\log^6 2n \right\rceil. \tag{6}$$

Let N satisfy that for all $n > N$ (6) holds and in addition $m < \frac{1}{2}n$. We easily get

$$\alpha_n > c \min_{m \leq N} \alpha_m > cN^{-3/8} \quad \forall n.$$

and we are done. □

Proof of the theorem. This is an immediate corollary from lemma 6. Take $\mathcal{V} = \emptyset$ and $E(0) = (n+1, 0)$ (so that $E(n)_1 > 0$ always) and then translate by $-n-1$. We get with probability $> 1 - Ce^{-c\log^2 n}$,

$$E(2n) > E(n) + cn \geq R(n) + cn$$

and since with probability $> 1 - Ce^{-c\log^2 n}$ we have $R(n) > -\log n\sqrt{n}$ the theorem is proved. □

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